INTERNATIONAL BACCALAUREATE
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MARKSCHEME

May 1999

MATHEMATICAL METHODS

Standard Level

Paper 2

20 pages

1. (a) $3x+4y=15$

(i) $A\left(0,\frac{15}{4}\right)$

$B(5,0)$

Note: Award *(A1)* for $\frac{15}{4}$, *(A1)* for 5, *(A1)* for both coordinates.

(ii) $|\Delta OAB| = \frac{1}{2}\left(\frac{15}{4}\right)(5)$ ***(M1)***

$= \frac{75}{8}$ (or equivalent *e.g.* $9\frac{3}{8}, 9.375, 9.38$ (3 s.f.)) ***(A1)***

(iii) $AB = \sqrt{\left(\frac{15}{4}\right)^2 + 5^2}$ ***(M1)***

$= \sqrt{39.0625} \quad \left(= \sqrt{\frac{625}{16}}\right)$

$= 6.25 \quad \left(= \frac{25}{4}\right)$ ***(A1)***

(b) (i) $|\Delta OAB| = \frac{1}{2} \times AB \times OC$ ***(M1)***

$\Rightarrow \frac{75}{8} = \frac{1}{2} \times \frac{25}{4} \times OC$

$\Rightarrow OC = 3$ ***(A2)***

OR (AB): $3x+4y=15$
(OC): $4x-3y=0$

$25x = 45$ ***(M1)***

$x = \frac{9}{5}$

$y = \frac{12}{5}$ ***(A1)***

$OC = \sqrt{\left(\frac{9}{5}\right)^2 + \left(\frac{12}{5}\right)^2} = 3$ ***(A1)***

continued...

Question 1 continued

(ii) $$\frac{|\Delta OCB|}{|\Delta OAB|}=\frac{OC^2}{OA^2}$$ *(M1)*

$$\Rightarrow |\Delta OCB| = \frac{75}{8}\times\frac{9}{225/16}$$ *(A1)*

$$=\frac{75}{8}\times\frac{9}{1}\times\frac{16}{225}$$ *(A1)*

$$=6$$ *(A1)*

Accept alternative methods, e.g.

$$CB=\sqrt{\left(5-\frac{9}{5}\right)^2+\left(\frac{12}{5}\right)^2}=4$$ *(M1)(A1)*

Therefore $\Delta=\frac{1}{2}\times 4\times 3=6$ *(A1)(A1)*

OR $$|\Delta OCB|=\frac{1}{2}\times 5\times 3\times\sin C\hat{O}B$$ *(M1)*

$$=\frac{15}{2}\times\cos O\hat{B}A$$ *(A1)*

$$=\frac{15}{2}\times\frac{5}{25/4}$$ *(A1)*

$$=\frac{15}{2}\times\frac{5}{1}\times\frac{4}{25}=6$$ *(A1)*

(iii) Method of finding AC may depend on methods used earlier in question, e.g.

$AC = AB - CB$ *(M1)*

$$=6\frac{1}{4}-4=2\frac{1}{4}$$ *(A1)*

OR $$|\Delta AOC|=|\Delta OAB|-|\Delta OCB|$$

$$=\frac{75}{8}-6=\frac{27}{8}$$ *(M1)*

Then $\frac{27}{8}=\frac{1}{2}(OC)\,AC$

$$=\frac{1}{2}\times 3\,AC\Rightarrow AC=\frac{9}{4}$$ *(A1)*

2. (a) (i) $f(x) = \dfrac{2x+1}{x-3}$

$= 2 + \dfrac{7}{x-3}$ by division or otherwise *(M1)*

Therefore as $|x| \to \infty \quad f(x) \to 2$ *(A1)*

$\Rightarrow y = 2$ is an asymptote *(AG)*

OR $\lim\limits_{x \to \infty} \dfrac{2x+1}{x-3} = 2$ *(M1)(A1)*

$\Rightarrow y = 2$ is an asymptote *(AG)*

OR make x the subject

$yx - 3y = 2x + 1$ *(M1)*

$x(y-2) = 1 + 3y$

$x = \dfrac{1+3y}{y-2}$ *(A1)*

$\Rightarrow \quad y = 2$ is an asymptote *(AG)*

Note: Accept inexact methods based on the ratio of the coefficients of x.

(ii) Asymptote at $x = 3$ *(A1)*

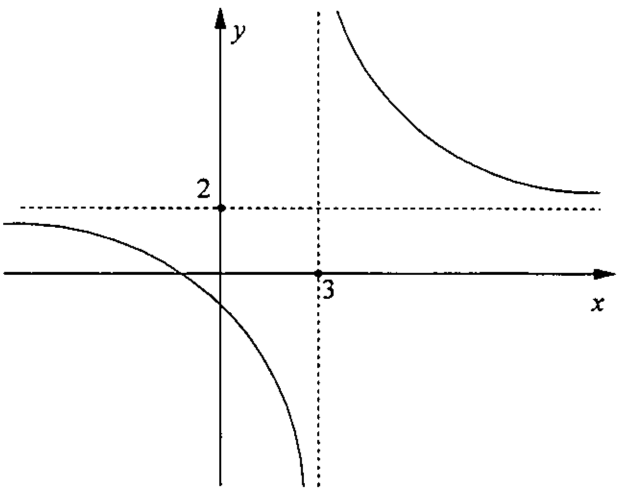
(iii) $P(3, 2)$ *(A1)*

(b) $f(x) = 0 \Rightarrow x = -\frac{1}{2} \quad \left(-\frac{1}{2}, 0\right)$ *(M1)(A1)*

$x = 0 \Rightarrow f(x) = -\frac{1}{3} \quad \left(0, -\frac{1}{3}\right)$ *(M1)(A1)*

Note: These do not have to be in coordinate form.

(c)



(A4)

Note:	
Asymptotes	*(A1)*
Intercepts	*(A1)*
"Shape"	*(A2)*

continued...

Question 2 continued

(d) $f'(x) = \dfrac{(x-3)(2)-(2x+1)}{(x-3)^2}$ *(M1)*

$= \dfrac{-7}{(x-3)^2}$ *(A1)*

= Slope at any point

Therefore slope when $x = 4$ is -7 *(A1)*
And $f(4) = 9$ i.e. $S(4, 9)$ *(A1)*
$\Rightarrow$ Equation of tangent: $y - 9 = -7(x-4)$ *(M1)*
$7x + y - 37 = 0$ *(A1)*

(e) at T, $\dfrac{-7}{(x-3)^2} = -7$ *(M1)*

$\Rightarrow (x-3)^2 = 1$ *(A1)*
$x - 3 = \pm 1$ *(A1)*
$\left.\begin{array}{l} x = 4 \text{ or } 2 \\ y = 9 \text{ or } -5 \end{array}\right\} \begin{array}{l} S(4, 9) \\ T(2, -5) \end{array}$ *(A1)(A1)*

(f) Midpoint $[ST] = \left(\dfrac{4+2}{2}, \dfrac{9-5}{2}\right)$

$= (3, 2)$

= point P *(A1)*

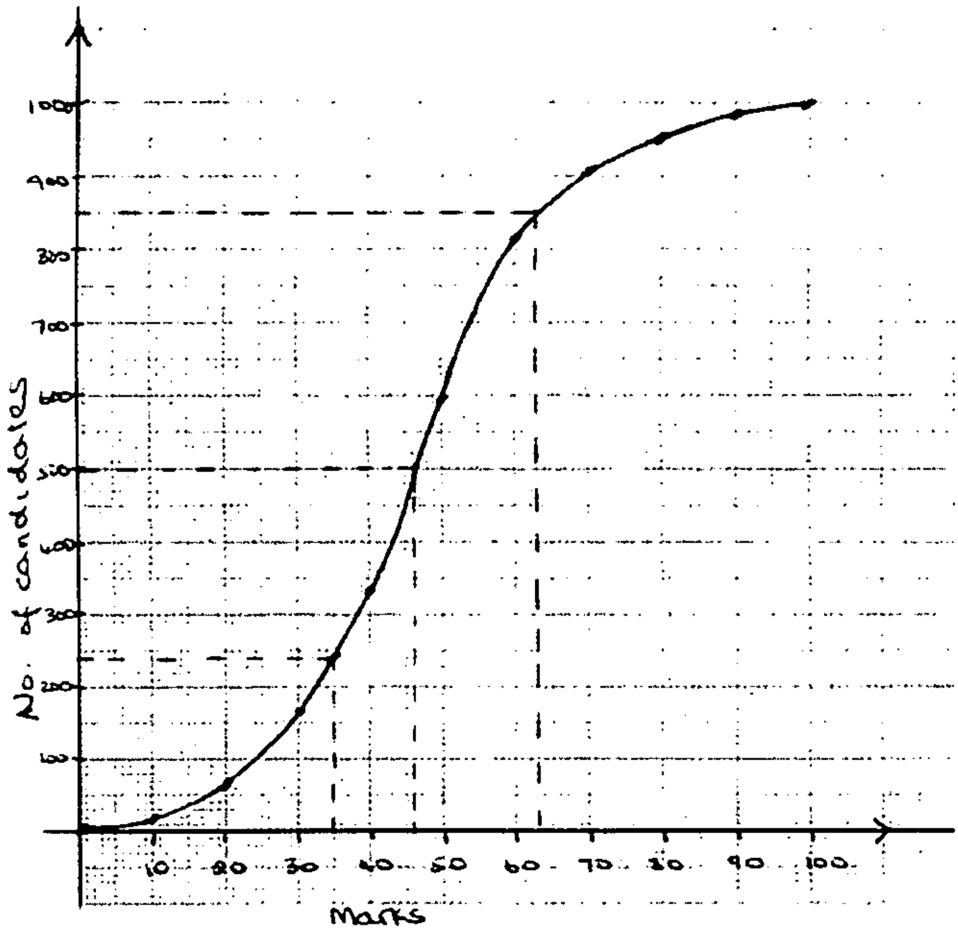
3. (a)

Mark	≤ 10	≤ 20	≤ 30	≤ 40	≤ 50	≤ 60	≤ 70	≤ 80	≤ 90	≤ 100
No. of Candidates	15	65	**165**	335	595	815	905	950	980	**1000**

(A3)

Note:	Award *(A1)* for 165, *(A1)* for 1000, *(A1)* if all other entries are correct.

(b)



(A5)

Notes:		
	Vertical axis and scale	*(A1)*
	Horizontal axis and scale	*(A1)*
	Points	*(A1)*
	Curve (allow polygon)	*(A2)*

(c) (i) Median = 46 *M1)(A1)*

(ii) Scores < 35: 240 candidates *(M1)(A2)*

(iii) Top 15 % $\Rightarrow$ Mark ≥ 63 *(M1)(A1)(A1)*

Note:	Accept the answers from the **student's** graph. In each part, award *(M1)* for the dotted lines on the graph.

4. (i) (a) $|\vec{OA}| = 6 \quad \Rightarrow$ A is on the circle. *(A1)*

$|\vec{OB}| = 6 \quad \Rightarrow$ B is on the circle. *(A1)*

$$|\vec{OC}| = \left|\begin{pmatrix} 5 \\ \sqrt{11} \end{pmatrix}\right|$$

$$= \sqrt{25+11}$$

$= 6 \quad \Rightarrow$ C is on the circle. *(A1)*

(b) $\vec{AC} = \vec{OC} - \vec{OA}$

$$= \begin{pmatrix} 5 \\ \sqrt{11} \end{pmatrix} - \begin{pmatrix} 6 \\ 0 \end{pmatrix}$$ *(M1)*

$$= \begin{pmatrix} -1 \\ \sqrt{11} \end{pmatrix}$$ *(A1)*

(c) $$\cos O\hat{A}C = \frac{\vec{AO} \cdot \vec{AC}}{|AO||AC|}$$ *(M1)*

$$= \frac{\begin{pmatrix} -6 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ \sqrt{11} \end{pmatrix}}{6\sqrt{1+11}}$$

$$= \frac{6}{6\sqrt{12}}$$ *(A1)*

$$= \frac{1}{2\sqrt{3}} = \frac{\sqrt{3}}{6}$$ *(A1)*

OR $$\cos O\hat{A}C = \frac{6^2 + (\sqrt{12})^2 - 6^2}{2 \times 6 \times \sqrt{12}}$$ *(M1)(A1)*

$$= \frac{1}{\sqrt{12}} \text{ as before}$$ *(A1)*

OR using the triangle formed by $\vec{AC}$ and its horizontal and vertical components:

$$\left|\vec{AC}\right| = \sqrt{12}$$ *(A1)*

$$\cos O\hat{A}C = \frac{1}{\sqrt{12}}$$ *(M1)(A1)*

Note: The answer is 0.289 to 3 s.f.

continued...

Question 4 continued

(d) A number of possible methods here

$\vec{BC} = \vec{OC} - \vec{OB}$

$= \begin{pmatrix} 5 \\ \sqrt{11} \end{pmatrix} - \begin{pmatrix} -6 \\ 0 \end{pmatrix}$ *(A1)*

$= \begin{pmatrix} 11 \\ \sqrt{11} \end{pmatrix}$ *(A1)*

$|BC| = \sqrt{132}$

$|\Delta ABC| = \frac{1}{2} \times \sqrt{132} \times \sqrt{12}$ *(A1)*

$= 6\sqrt{11}$ *(A1)*

OR ΔABC has base $AB = 12$ *(A1)*

and height $= \sqrt{11}$ *(A1)*

$\Rightarrow$ area $= \frac{1}{2} \times 12 \times \sqrt{11}$ *(A1)*

$= 6\sqrt{11}$ *(A1)*

OR Given $\cos B\hat{A}C = \frac{\sqrt{3}}{6}$

$\sin B\hat{A}C = \frac{\sqrt{33}}{6} \Rightarrow |\Delta ABC| = \frac{1}{2} \times 12 \times \sqrt{12} \times \frac{\sqrt{33}}{6}$ *(A1)(A1)(A1)*

$= 6\sqrt{11}$ *(A1)*

(ii) (a) $M^2 = \begin{pmatrix} a & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} a & 2 \\ 2 & -1 \end{pmatrix}$

$= \begin{pmatrix} a^2 + 4 & 2a - 2 \\ 2a - 2 & 5 \end{pmatrix}$ *(A1)(A1)(A1)(A1)*

(b) $2a - 2 = -4$

$\Rightarrow a = -1$ *(A1)*

Substituting: $a^2 + 4 = (-1)^2 + 4 = 5$ *(A1)*

Note:	Candidates may solve $a^2 + 4 = 5$ to give $a = \pm 1$, and then show that only $a = -1$ satisfies $2a - 2 = -4$.

continued...

Question 4 continued

(c) $M = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}$

$M^{-1} = -\frac{1}{3}\begin{pmatrix} -1 & -2 \\ -2 & -1 \end{pmatrix}$ ***(M1)***

$= \frac{1}{3}\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1/3 & 2/3 \\ 2/3 & 1/3 \end{pmatrix}$ ***(A1)***

$-x + 2y = -3$
$2x - y = 3$

$\Rightarrow \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \end{pmatrix}$ ***(M1)(M1)***

$\Rightarrow \begin{pmatrix} 1/3 & 2/3 \\ 2/3 & 1/3 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/3 & 2/3 \\ 2/3 & 1/3 \end{pmatrix}\begin{pmatrix} -3 \\ 3 \end{pmatrix}$ ***(A1)***

$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ ***(A1)***

i.e. $x = 1$
$y = -1$

Note: The solution must use matrices. Award no marks for solutions using other methods.

5. (i) (a)

$$f''(x) = 2x - 2$$

$$\Rightarrow f'(x) = x^2 - 2x + c \qquad \textit{(M1)(M1)}$$

$$= 0 \text{ when } x = 3$$

$$\Rightarrow 0 = 9 - 6 + c$$

$$c = -3 \qquad \textit{(A1)}$$

$$f'(x) = x^2 - 2x - 3 \qquad \textit{(AG)}$$

$$f(x) = \frac{x^3}{3} - x^2 - 3x + d \qquad \textit{(M1)}$$

When $x = 3, f(x) = -7$

$$\Rightarrow -7 = 9 - 9 - 9 + d \qquad \textit{(M1)}$$

$$\Rightarrow d = 2 \qquad \textit{(A1)}$$

$$\Rightarrow f(x) = \frac{x^3}{3} - x^2 - 3x + 2$$

(b)

$$f(0) = 2 \qquad \textit{(A1)}$$

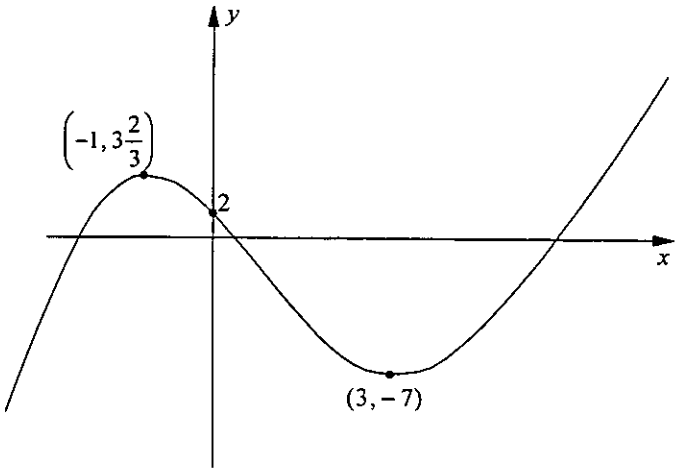
$$f(-1) = -\frac{1}{3} - 1 + 3 + 2$$

$$= 3\frac{2}{3} \qquad \textit{(A1)}$$

$$f'(-1) = 1 + 2 - 3$$

$$= 0 \qquad \textit{(A1)}$$

(c) $f'(-1) = 0 \Rightarrow \left(-1, 3\frac{2}{3}\right)$ is a stationary point



(A4)

Notes: Award *(A1)* for maximum, *(A1)* for $(0, 2)$ *(A1)* for $(3, -7)$, *(A1)* for cubic

continued…

Question 5 continued

(ii) (a) $y = e^{x/2}$ at $x = 0$ $y = e^0 = 1$ $P(0, 1)$ *(A1)(A1)*

(b) (i) $V = \pi\int_0^{\ln 2}(e^{x/2})^2\,dx$ *(A4)*

Notes:	Award *(A1)* for π
	(A1) for each limit
	(A1) for $(e^{x/2})^2$

(ii) $V = \pi\int_0^{\ln 2} e^x dx$ *(A1)*

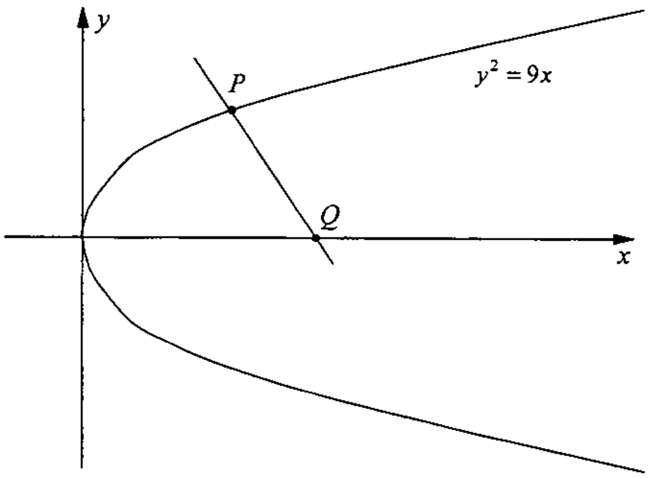
$= \pi\left[e^x\right]_0^{\ln 2}$ *(A1)*

$= \pi\left[e^{\ln 2} - e^0\right]$ *(A1)*

$= \pi[2 - 1] = \pi$ *(A1)(A1)*

$= \pi$ *(AG)*

(iii) (a)



$y^2 = 9x$

$6^2 = 9(4)$ *(M1)*

$36 = 36$ *(A1)*

$\Rightarrow$ (4, 6) on parabola

Question 5 continued

(b) (i) $y = 3\sqrt{x}$

$\dfrac{dy}{dx} = \dfrac{3}{2\sqrt{x}}$ *(M1)*

= Slope at any point

Therefore at $(4, 6)$, slope of tangent $= \dfrac{3}{4}$ *(A1)*

$\Rightarrow$ Slope of normal $= -\dfrac{4}{3}$ *(A1)*

Therefore equation of normal is $y - 6 = -\dfrac{4}{3}(x - 4)$ *(M1)*

$3y - 18 = -4x + 16$

$4x + 3y - 34 = 0$ *(A1)*

> **Notes:** Candidates may differentiate implicitly to obtain $\dfrac{dy}{dx} = \dfrac{9}{2y}$.
> Answer must be given in the form $ax + by + c = 0$.

(ii) Coordinates of Q:

$y = 0, 4x = 34$

$x = \dfrac{17}{2}$ *(A1)*

$Q\left(\dfrac{17}{2}, 0\right)$ *(A1)*

(c) $SP = \sqrt{\left(\dfrac{9}{4} - 4\right)^2 + (0 - 6)^2}$ *(M1)*

$= \sqrt{\dfrac{49}{16} + 36}$

$= \dfrac{25}{4}$ *(A1)*

$SQ = \dfrac{17}{2} - \dfrac{9}{4}$ *(M1)*

$= \dfrac{34}{4} - \dfrac{9}{4}$

$= \dfrac{25}{4}$ *(A1)*

continued...

Question 5 continued

(d) $|SP| = |SQ| \Rightarrow S\hat{P}Q = S\hat{Q}P$ *(M1)*

But $S\hat{Q}P = M\hat{P}Q$ (alternate angles) *(A1)*

$\Rightarrow M\hat{P}Q = S\hat{P}Q$ *(A1)*

6. (i) $p(\text{Red}) = \frac{35}{40} = \frac{7}{8}$ $\quad p(\text{Black}) = \frac{5}{40} = \frac{1}{8}$

(a) (i) $p(\text{one black}) = \binom{8}{1}\left(\frac{1}{8}\right)^1\left(\frac{7}{8}\right)^7$ *(M1)(A1)*

$= 0.393$ to 3 s.f. *(A1)*

(ii) $p(\text{at least one black}) = 1 - p(\text{none})$ *(M1)*

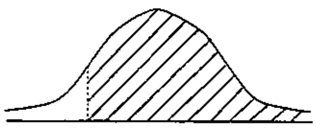
$= 1 - \binom{8}{0}\left(\frac{1}{8}\right)^0\left(\frac{7}{8}\right)^8$ *(A1)*

$= 1 - 0.344$
$= 0.656$ *(A1)*

(b) (i) 400 draws: expected number of blacks $= \frac{400}{8}$ *(M1)*
$= 50$ *(A1)*

(ii) $\mu = 50$
$\sigma = \sqrt{400 \times \frac{1}{8} \times \frac{7}{8}}$
$= 2.5\sqrt{7}$
$= 6.614...$ *(M1)(A1)*

(a) $p(\text{at least 48}) = p(x \geq 47.5)$ *(A1)*
$= p\left(z \geq \frac{50 - 47.5}{6.614...}\right)$
$= p(z \geq 0.378)$ (3 s.f.) *(A1)*
≈ 0.647 (3 s.f.) *(A1)*



(b) $p(\text{exactly 48}) = p(47.5 \leq x \leq 48.5)$ *(M1)*
$= 0.647 - p\left(z \geq \frac{1.5}{6.614...}\right)$ *(A1)*
$= 0.647 - p(z \geq 0.227)$ *(A1)*
$= 0.647 - 0.590$ (3 s.f.) *(A1)*
$= 0.057$ *(A1)*

Note: Accept any answer between 0.057 and 0.058.
In marking, be generous in interpretation of significant figures but do not accept more than three. Deduct for absurd levels of "accuracy" obtained from a calculator, but this may be the only situation.

continued...

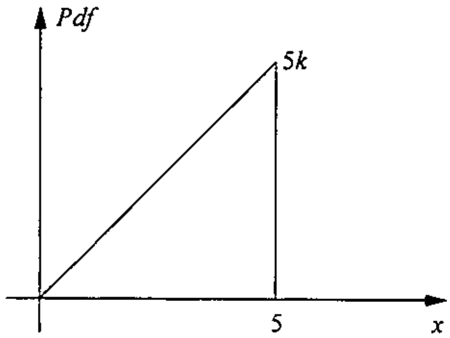
Question 6 continued

(ii) (a) $\int_0^5 kx\,\mathrm{d}x = 1$ *(M1)*

$\Rightarrow \left[\frac{kx^2}{2}\right]_0^5 = 1$ *(A1)(A1)*

$\Rightarrow k = \frac{2}{25} = 0.08$ *(A1)*

OR Area = 1 *(M1)*



$\Rightarrow \frac{1}{2} \times 5 \times 5k = 1$ *(A1)(A1)*

$\Rightarrow k = \frac{2}{25}$ *(A1)*

(b) $E(X) = \int_0^5 x f(x)\,\mathrm{d}x$ *(M1)*

$= \left[\frac{2}{25} \times \frac{x^3}{3}\right]_0^5$ *(A1)*

$= \frac{2}{25} \times \frac{125}{3}$ *(A1)*

$= \frac{10}{3} = 3.33$ *(A1)*

(c) $\text{Var}(X) = E(X^2) - [E(X)]^2$ *(M1)*

$= \int_0^5 x^2 kx\,\mathrm{d}x - \left(\frac{10}{3}\right)^2$ *(A1)*

$= \left[\frac{2}{25} \times \frac{x^4}{4}\right]_0^5 - \frac{100}{9}$ *(A1)*

$= \frac{25}{2} - \frac{100}{9}$

$= \frac{25}{18} = 1.39$ *(A1)*

continued...

Question 6 continued

(d) $p(2 \le x \le 3) = \int_2^3 f(x)\,dx$ *(M1)*

$= \left[\frac{2}{25} \times \frac{x^2}{2}\right]_2^3$ *(A1)*

$= \frac{1}{25}[9-4]$ *(A1)*

$= 0.2$ *(A1)*

(e) Let the median be m

Then $p(X \le m) = 0.5$ *(M1)*

$\Rightarrow \int_0^m \frac{2}{25}x\,dx = \frac{1}{2}$ *(M1)*

$\frac{m^2}{25} = \frac{1}{2}$ *(A1)*

$m^2 = \frac{25}{2}$

$m = \pm\frac{5\sqrt{2}}{2}$ *(A1)*

But m cannot be negative since $0 \le x \le 5$ *(A1)*

$\Rightarrow m = \frac{5\sqrt{2}}{2} = 3.54$ *(A1)*